

# Particle renormalizations in presence of dissipative environments

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We study the Aharonov-Bohm oscillations of a charged particle on a ring of radius  $R$  coupled to a dirty metal environment. With Monte-Carlo methods we evaluate the curvature of these oscillations which has the form  $1/M^*R^2$ , where  $M^*$  is an effective mass. We find that at low temperatures  $T$  the curvature approaches at large  $R > l$  an  $R$  independent  $M^* > M$ , where  $l$  is the mean free path in the metal. This behavior is also consistent with perturbation theory in the particle - metal coupling parameter. At finite temperature  $T$  we identify dephasing lengths that scale as  $T^{-1}$  at  $R \gtrsim l$  and as  $T^{-1/4}$  at  $R \ll l$ .

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## I. INTRODUCTION

The problem of interference in presence of a dissipative environment is fundamental for a variety of experimental systems. Interference has been monitored by Aharonov-Bohm (AB) oscillations in mesoscopic rings<sup>1,2,3</sup> or in quantum Hall edge states<sup>4</sup> in presence of noise from gates or other metal surfaces. Cold atoms trapped by an atom chip are sensitive to the noise produced by the chip<sup>5,6,7</sup>. In particular giant Rydberg atoms are studied<sup>8</sup> whose huge electric dipole is highly susceptible to such noise.

An efficient tool for monitoring the effect of the environment, as proposed by Guinea<sup>9</sup>, is to find the AB oscillation amplitude as function of the radius  $R$  of the ring. This amplitude is measured by the curvature<sup>10,11,12</sup> of the ground state energy  $E_0$  at external flux  $\phi_x = 0$ , i.e.  $1/M^*R^2 = \partial^2 E_0 / \partial \phi_x^2|_0$ , defining an effective mass  $M^*$ . For free particles of mass  $M$  this curvature is the mean level spacing  $1/MR^2$ . The particle can be coupled to a variety of environments, with three systems of particular interest: (i) a Caldeira-Legget (CL) bath<sup>9</sup>, (ii) a charged particle in a dirty metal environment<sup>9,13</sup> and (iii) a particle with an electric dipole in a dirty metal environment<sup>14</sup>. System (i) has been studied with a large variety of methods, all showing that the AB amplitude is exponentially suppressed  $\sim e^{-\pi^2 \gamma R^2}$ , i.e. a new length scale  $\sim 1/\sqrt{\gamma}$  is generated by the coupling  $\gamma$  to the environment<sup>9</sup>. System (ii) has been studied by renormalization group (RG) methods<sup>9,15</sup> finding  $M^* \sim R^\mu$  with a small  $\mu$ , a Monte Carlo (MC) numerical method gave<sup>13</sup>  $\mu = 1.8$  at sufficiently large  $R$ , while a variational scheme<sup>14</sup> gave  $\mu = 0$ . System (iii) was also studied within the variational scheme<sup>14</sup>, leading to  $\mu = 0$  as well.

In the present work we use MC methods to analyze mostly system (ii). We find that the energy cutoff used in a previous study<sup>13</sup> is insufficient and a higher cutoff  $\omega_c$  is needed. In particular we find that at large  $R > l$  the effective mass  $M^*$  is  $R$  independent, i.e.  $\mu = 0$ , where  $l$  is the mean free path in the metal. For  $R > l$  we also find that at temperature  $T$  the data scales as  $TR$ , identifying a length scale  $\sim 1/T$ . For  $R \ll l$  the system reduces to a CL one with a  $\sim T^{-1/4}$  length scale.

A non-equilibrium study<sup>16</sup> has found dephasing lengths that have the same power laws, establishing a connection between equilibrium and non-equilibrium results.

## II. THE MODEL

The time dependent angular position  $\theta_m(\tau)$  of a particle on the ring has in general a winding number  $m$  so that  $\theta_m(\tau) = \theta(\tau) + 2\pi m T \tau$  where  $\theta(0) = \theta(1/T)$  has periodic boundary condition. In presence of an external flux  $\phi_x$  (in units of the flux quantum  $hc/e$ ) the partition sum has the form

$$Z = \sum_m e^{2\pi i m \phi_x} \int \mathcal{D}\theta e^{-S^{(m)}} \\ S^{(m)} = \frac{1}{2} M R^2 \int_0^{1/T} \left( \frac{\partial \theta}{\partial \tau} + 2\pi m T \right)^2 d\tau + \\ \alpha \int_0^{1/T} \int_0^{1/T} \frac{\pi^2 T^2 K[\theta(\tau) - \theta(\tau') + 2\pi m T(\tau - \tau')]}{\sin^2 \pi T(\tau - \tau')} d\tau d\tau' \quad (1)$$

where the effect of environments, in each of the 3 cases, is<sup>9,13,14</sup>

$$\begin{aligned} K(z) &= \sin^2 z/2; & \alpha &= \gamma R^2 & (i) \\ &= 1 - [4r^2 \sin^2 \frac{z}{2} + 1]^{-1/2}; & \alpha &= \frac{3}{8k_F^2 l^2} & (ii) \\ &= 1 - [4r^2 \sin^2 \frac{z}{2} + 1]^{-3/2}; & \alpha &= \frac{p^2}{e^2 l^2} \frac{3}{8k_F^2 l^2} & (iii). \end{aligned} \quad (2)$$

Case (i) is the CL system where  $\gamma$  is the coupling to a harmonic oscillator bath; case (ii) is a charge coupled to a dirty metal where  $k_F$  is the Fermi wavevector,  $l$  is the mean free path in the metal, and  $r = R/l$ ; case (iii) is an electric dipole of strength  $p$  coupled to a dirty metal.

We note that the forms (ii) and (iii) are based<sup>13,14</sup> on a wavevector and frequency dependent dielectric function for the metal of the form  $\epsilon(q, \omega) = 1 + 4\pi\sigma/(-i\omega + Dq^2)$  valid at  $q < 1/\ell$ , where  $\sigma$  is the conductivity and  $D$  is the diffusion constant of the metal. The  $q$  integrals are cutoff

by  $q < 1/\ell$ , hence the forms (ii) and (iii) are valid at  $r \gtrsim 1$ . We will use below these forms also at  $r < 1$  since they represent qualitatively the decrease of  $K(z)$  with  $r$ . Furthermore, at  $r \rightarrow 0$  the form (ii) reduces to that of the CL model (i) with  $\alpha_{CL} = 2\alpha r^2$ .

We also note that in model (ii)  $\alpha < 1$  for relevant metals. However, model (iii) allows for a large  $\alpha$  since the dipole parameter  $p$  can be large, as e.g. in the Rydberg atoms<sup>8</sup>.

We are interested in the effect of the environment on the visibility of quantum interference as measured by the particle. As a measure of this visibility we consider the curvature of the Aharonov-Bohm oscillations

$$\frac{1}{M^*(T)R^2} = \frac{\partial^2 F}{\partial \phi_x^2} \Big|_{\phi_x=0} \quad (3)$$

where  $F = -T \ln Z$ . It is useful to consider a free particle  $\alpha = 0$ , for which

$$\left( \frac{M}{M^*(T)} \right)_{\alpha=0} = 2\pi^2 t \sum_m m^2 e^{-\pi^2 m^2 t} / \sum_m e^{-\pi^2 m^2 t} \equiv f(t) \quad (4)$$

where  $t = 2MR^2T$ . This identifies the thermal length  $L_T \sim 1/\sqrt{MT}$ .

In the interacting system a high energy cutoff can be identified by considering  $\tau \rightarrow \tau'$  (corresponding to high frequencies  $\omega$ ) so that expansion of  $K(z)$  and Fourier transform yield

$$S^{(m)} \rightarrow \frac{1}{2} \int \frac{d\omega}{2\pi} [MR^2\omega^2 + 2\pi\alpha K''(0)|\omega|] |\theta(\omega)|^2 + (2\pi m)^2 [\frac{1}{2}MR^2T + \alpha K''(0)]. \quad (5)$$

The term linear in  $|\omega|$  is typical for dissipative systems, i.e. the environment induces dissipation on the particle. The cutoff  $\omega_c$  is now identified when the kinetic  $\sim \omega^2$  and  $\sim |\omega|$  interaction terms are comparable, i.e.

$$\omega_c = \frac{2\pi\alpha K''(0)}{MR^2}. \quad (6)$$

This  $\omega_c$  replaces a possibly higher environment cutoff, since significant renormalizations start only below  $\omega_c$  where the linear  $|\omega|$  dispersion leads to  $\ln \omega$  terms in perturbation theory and to the need for either RG treatment, or an equivalent variational scheme<sup>14</sup>. Note that  $K''(0) = \frac{1}{2}; r^2; 3r^2$  in the 3 models above, hence  $\omega_c = \pi\gamma/M$  in case (i), while  $\omega_c \sim \alpha/Ml^2$  in cases (ii) and (iii).

### III. MONTE CARLO PROCEDURE

For the MC numerical method we need to discretize the time axis into a Trotter number  $N_T$  of segments, i.e. the time interval of each segment is  $\Delta\tau = 1/(TN_T)$ . The discrete action is

$$S^{(m)} = \frac{1}{2} [MR^2 N_T T + \alpha K''(0)] \sum_n (\theta_{n+1} - \theta_n + \frac{2\pi m}{N_T})^2$$

$$+ \frac{\alpha\pi^2}{N_T^2} \sum_{n \neq n'} \frac{K(\theta_n - \theta_{n'} + 2\pi m(n - n')/N_T)}{\sin^2(\pi(n - n')/N_T)}. \quad (7)$$

The  $\frac{1}{2}\alpha K''(0)$  term comes from the  $n = n'$  interaction term by expanding  $K(z)$  around  $z = 0$ . A key issue in our MC study is the choice of energy cutoff  $1/\Delta\tau$  and the corresponding Trotter number  $N_T = 1/(T\Delta\tau)$ . The correct choice is such that the free kinetic term dominates over the single  $n = n'$  interaction term, i.e.  $N_T \gtrsim \omega_c/T$ , with  $\omega_c$  from Eq. (6). Hence  $\Delta\tau \approx 1/\omega_c$  corresponds to the cutoff  $\omega_c$  as identified by RG or variational methods. A previous MC study on the charge problem<sup>13</sup> has chosen  $N_T$  in the range  $1/t$  to  $4/t$ , i.e. an energy cutoff of  $\approx 1/MR^2$ . For large  $r$  this cutoff is much smaller than  $\omega_c$  and is therefore insufficient.

Eqs. (1,3) identify  $1/M^*(T)R^2 = 2\pi^2 T \langle m^2 \rangle|_{\phi_x=0}$  so that the MC evaluates the fluctuations in winding number  $\langle m^2 \rangle$  at external flux  $\phi_x = 0$ . The procedure is to start with some  $m$ , update  $\theta_n$  at a time position  $n$  to  $\theta'_n$  and accept or reject the change according to the MC rule with probability  $\exp[S^{(m)}\{\theta_n\} - S^{(m)}\{\theta'_n\}]$ . After the  $N_T$  points are successively updated, the winding number is shifted to  $m' = m \pm 1$  and the shift is accepted or rejected with the probability  $\exp[S^{(m)}\{\theta_n\} - S^{(m')}\{\theta_n\}]$ . An update of  $\theta_n$  is done randomly with a step size that produces an acceptance ratio of about 50%<sup>11</sup>.

The inset in Fig. 1 shows the  $N_T$  dependence of  $M/M^*$  for the charge problem with  $r = 5, t = 0.2, \alpha = 0.019$ . A choice for  $N_T$  in the range  $1/t - 4/t$  is clearly insufficient; saturation sets in around  $N_T \approx 100$  which is of order of  $\omega_c/T = 30$ . In the following we choose our  $N_T$ , in the charge problem, to be  $N_T = 40\alpha r^2/t = 10\omega_c/(\pi T)$ , i.e.  $N_T = 95$  for the inset parameters. For the dipole case, where  $\omega_c$  is 3 times higher we choose  $N_T = 120\alpha r^2/t = 10\omega_c/(\pi T)$ . Fig. 1 shows that for  $r = 5, t = 0.2, \alpha = 0.02$  (red squares) saturation indeed sets in near  $N_T = 300$ .

This high value of  $N_T$  restricts realistic MC studies. We have noticed, however, that this high  $N_T$  is necessary only in the vicinity of  $n = n'$  in the double sum of (7), where the summand is rapidly varying. Hence the double sum is taken over all points only in the vicinity of the singularity, i.e. for  $|n - n'| < 0.03N_T$ . For points that are further separated we coarse grain the sum with fewer points, corresponding to an effective  $N_T = 1/t$ .

The results of this procedure are shown by the green circles in Fig. 1, and are in agreement with the full calculation that includes all  $N_T$  points. The double sum has then  $\approx \frac{1}{2}10^{-3}N_T^2 + \frac{1}{2}t^{-2}$  terms, much less than the  $\frac{1}{2}N_T^2$  terms of the full calculation. We also show data where the double sum is coarse grained at all points, including those near  $n = n'$ , by blue triangles. Here the double sum has only  $\frac{1}{2}t^{-2}$  terms; this data has significant deviations from the full calculation.

We proceed to discuss our error estimates. At low temperatures we evaluate  $\langle m^2 \rangle$ , and the average involves typically many values of  $m$ . To estimate errors we evaluate the correlation function for a given run and deduce a correlation length  $\xi$ . We discard the initial  $10^4$  MC iter-

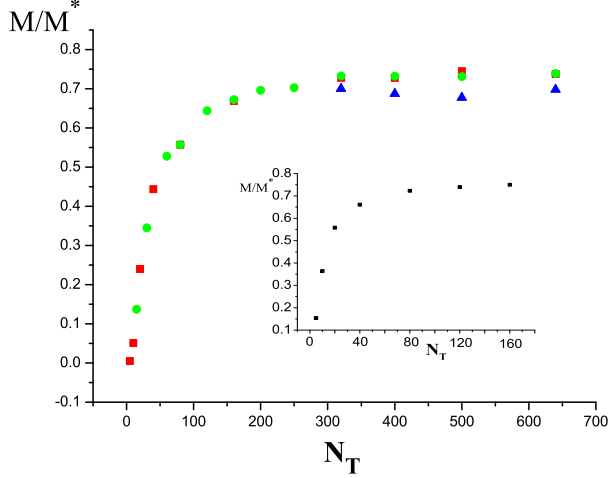


FIG. 1: Trotter number dependence of the effective mass for the dipole case with  $r = 5, t = 0.2, \alpha = 0.02$ , using (i) all  $N_T$  points in the double sum Eq. (7) – red squares, (ii) For points  $|n - n'| > 0.03N_T$  sum is coarse grained (see text) – green circles, (iii) the whole sum is coarse grained – blue triangles. Inset: The charge case with  $r = 5, t = 0.2, \alpha = 0.019$  using all  $N_T$  points in the sums.

ations and then evaluate the standard deviation  $\sigma$  of the average data; the error is then<sup>17</sup>  $\sigma\sqrt{2\xi + 1}$ . We typically find a short correlation length of a few units and we run till an error of  $\sim 2\%$  is achieved; the number of iterations is then  $\approx (1 - 2) \cdot 10^5$  and in some cases up to  $10^6$ , where each iteration is an update of  $N_T$  values of the  $\theta_n$ .

At high temperatures  $t > 1$ , where  $M/M^* \lesssim 10^{-3}$ , the probability of  $m \neq 0$  becomes extremely small so that just  $m = \pm 1$  determine the outcome<sup>11</sup>. Hence we evaluate  $\langle m^2 \rangle = 2\langle e^{S_1 - S_0} \rangle_0$ , averaging with  $e^{-S_0}$ . In this method we find a rather long correlation length of  $\sim 10^3$ , yet there is no need to vary  $m$  and a 2% accuracy can be achieved after  $\approx (1 - 2) \cdot 10^5$  iterations.

#### IV. MC RESULTS

We present here our data for the dirty metal, system (ii). In Fig. 2 we show our data for  $\alpha = 0.019$  at low temperatures,  $t < 0.3$ ; we note saturation at  $t < 0.2$ . In Fig. 3 we collect the limiting low  $t$  values of our data for various  $\alpha$ , typically achieved at  $t \approx 0.1 - 0.01$ . The data is limited to Trotter numbers  $N_T = 40\alpha r^2/t < 9000$ .

We compare in Fig. 3 the data with results of perturbation theory (Appendix I). The perturbation is formally first order in  $\alpha$ , however, it should be valid also for large  $\alpha$  and small  $r$  such that  $x \lesssim 2$ , where at  $t = 0$  we define  $x = M^*(t = 0)/M$ . The perturbation curves are a good fit to the data for  $r \lesssim 1$ , while at  $r > 1$  and small  $\alpha$  the

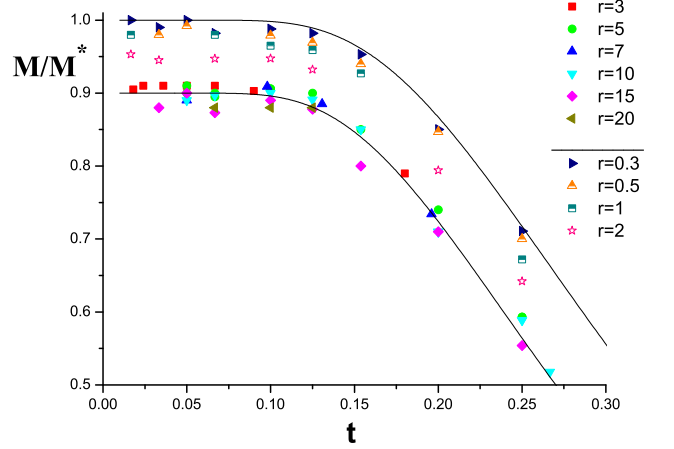


FIG. 2: AB curvature as function of reduced temperature with  $\alpha = 0.019$ . All  $r \geq 3$  values fit the renormalized form  $0.9f(t/0.9)$  – the lower curve. At  $r \leq 1$  the data approaches  $f(t)$  of a free particles – the upper curve.

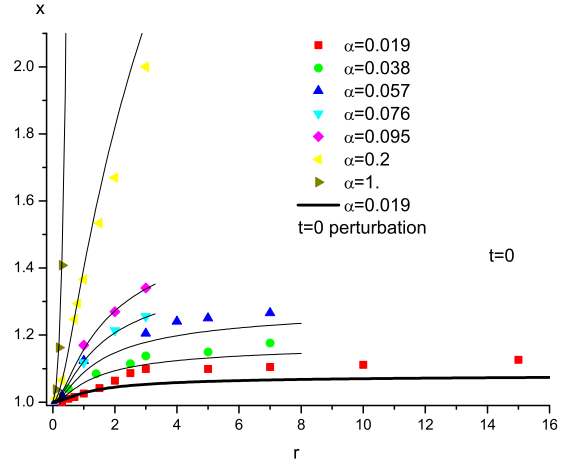


FIG. 3:  $t = 0$  limiting values of  $x = M^*(t = 0)/M$  for various  $\alpha$ . The full lines are results of perturbation expansion (Appendix I).

fit is qualitatively good, in the sense that saturation is achieved at large  $r$ . We have also attempted to fit these data by a scaling function of the form  $x = 1 + r^{2-c}g(\alpha r^c)$ , that is consistent with the  $r \rightarrow 0$  form of the perturbation expansion. In particular, this form with  $c = 2$  would scale onto the CL system at  $r \rightarrow 0, \alpha \rightarrow \infty$ . However, we could not find a satisfactory fit even for the small  $r \lesssim 2$  regime.

Our data shows for the lowest  $\alpha = 0.019$  and for  $r \geq 3$

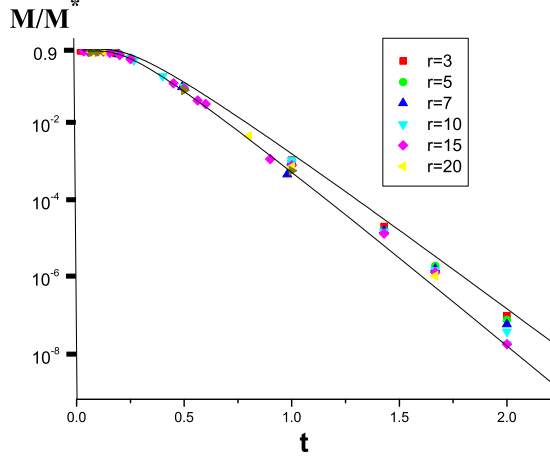


FIG. 4: AB curvature including high temperatures with  $\alpha = 0.019$ . All data fall in between the upper line  $f(t)$  and the lower line  $0.9f(t/0.9)$ .

that  $M/M^*$  reaches saturation with  $M/M^* \approx 0.9$ , almost independent of  $r$ . The data at  $r = 20$  (shown in Fig. 2) is consistent with this saturation, though it is not shown in Fig. 3 to keep a convenient scale. In view of this saturation at  $3 < r < 20$  we expect it to persist at higher  $r$ . In terms of  $M^* \sim r^\mu$ , our data shows that  $\mu \lesssim 0.05$  and is consistent with  $\mu = 0$ . We note that with our revised values of  $N_T$  we were not able to reach a saturation regime at larger  $\alpha$ , see Fig. 3.

Our result shows that the AB curvature  $\sim 1/R^2$  is the same as for free particles, i.e. the ground state has no anomaly, at least for weak  $\alpha = 0.019$ . Furthermore, Fig. 2 shows that  $M^*$  determines the finite temperature behavior, as long as  $T \ll \omega_c$ . Thus if we replace  $M \rightarrow M^* = M/0.9$  in Eq. (4) we obtain the lower curve  $0.9f(t/0.9)$  in Fig. 2 which is a good fit to the data. The thermal length is then  $L_T \sim 1/\sqrt{M^*T}$ .

In Fig. 4 we show our  $r \geq 3$  data up to  $t = 2$ . The data falls in between two lines:  $0.9f(t/0.9)$  and  $f(t)$ . The lower curve  $0.9f(t/0.9)$  corresponds to the renormalized system and fits data with  $T \ll \omega_c$ , i.e.  $t \ll 4\pi\alpha r^2$ . For a fixed  $t$  as  $r$  decreases  $T$  approaches  $\omega_c$  and the data approaches the upper curve which is the unrenormalized free particle form  $f(t)$ .

We therefore parameterize our data by a function  $x(r, t)$  such that  $M/M^* = f(tx)/x$ . In this way we avoid the obvious  $t$  dependence associated with mass renormalization and focus on additional temperature effects. In Fig. 5 we show that for  $r \gtrsim 1$  the data for  $x(t, r)$  scales with  $t/r$ . Since  $t \sim TR^2$  the scaling parameter is  $\sim TR$ , identifying a length scale  $\sim 1/T$ . A dephasing length scale has been recently derived in a non-equilibrium study<sup>16</sup> which for  $r \gtrsim 1$  indeed scales with  $1/T$ . We propose therefore that the additional  $T$  de-

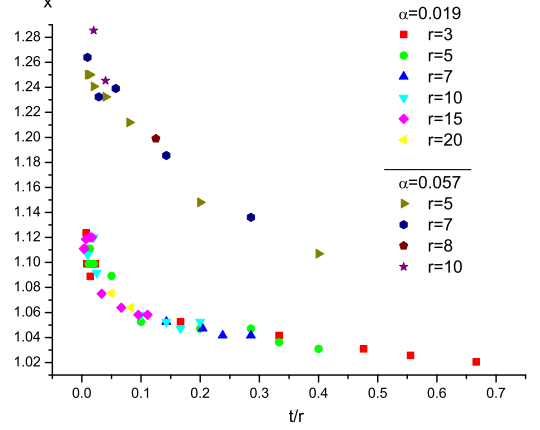


FIG. 5: Scaling of the  $x$  variable in  $M/M^* = f(tx)/x$  for  $r \gtrsim 1$  cases, with  $\alpha = 0.019$  and  $\alpha = 0.057$ .

pendence embedded in our variable  $x(t, r)$  is related to dephasing of the non-equilibrium situation.

We note that the perturbation expansion yields for  $r \gg 1$ ,

$$\frac{M}{M^*} = 1 - 4\alpha + O\left(\frac{\alpha t}{r} \ln r\right) \quad r \gg 1. \quad (8)$$

While the dependence on  $t/r$  is consistent with Fig. 5 (up to a  $\ln r$  factor), we note that the  $t/r$  form in the perturbation form (8) is valid only at  $t \ll 1$  and  $r \gtrsim 10$ . Hence the observed scaling, Fig. 5, with  $t/r$  up to  $t \approx 1$  and at  $3 < r < 20$  is an unexpected feature.

In Fig. 6 we show that for  $r \ll 1$  the data scales as  $tr^2$ . At  $tr^2 \lesssim 0.04$  both  $x(t, r)$  and  $x(0, r)$  are close to 1 and the errors in  $1/x(t, r) - 1/x(0, r)$  are too large to draw a conclusion in this regime. The same difficulty is with all data of small  $\alpha$ , hence Fig. 6 shows only  $\alpha = 0.2, 1$ . At  $tr^2 \gtrsim 0.04$  the data in Fig. 6 supports a  $tr^2$  scaling. Since  $t \sim TR^2$  this implies a length scale  $\sim T^{-1/4}$ . We note again that similar dependence for a dephasing length was found for  $r \ll 1$  in the non-equilibrium study<sup>16</sup>.

For  $r \ll 1$  we can use the perturbation result Eq. (A12)

$$\frac{M}{M^*} = 1 - 2\alpha \sum_n a_n + 4tar^2 \quad r \ll 1. \quad (9)$$

This shows the  $\alpha r^2$  scaling at  $tar^2 \ll 1$ . It is remarkable that our data in Fig. 6 supports  $\alpha r^2$  scaling up to rather high temperatures of  $t \lesssim 1$ .

As noted above, the  $r$  dependence of  $K(z)$  is reliable only at  $r \gtrsim 1$  where the low  $q, \omega$  form of  $\epsilon(q, \omega)$  can be used, or at  $r \ll 1$ , which is the CL limit. In fact, for a general  $\epsilon(q, \omega)$  one can expand the response in  $R$  and obtain that the leading term is  $K(z) \sim R^2$ , i.e. the CL form. We conclude then that at both small and large  $r$ ,

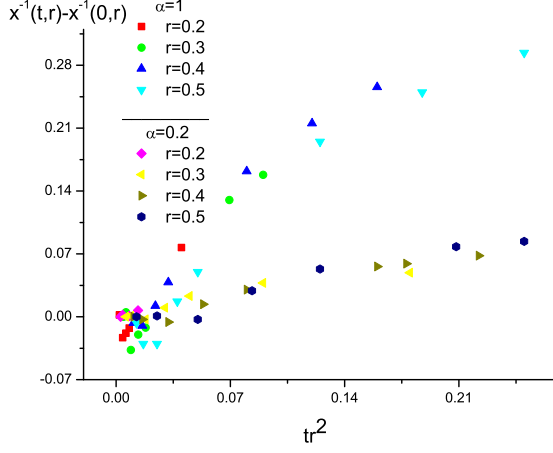


FIG. 6: Scaling of the variable  $\frac{1}{x(t,r)} - \frac{1}{x(0,r)}$  for  $r \ll 1$  cases, with  $\alpha = 0.2$  and  $\alpha = 1$ .

where  $K(z)$  is reliable, the  $T$  dependent length scale of the equilibrium observable  $M^*/M$  can be identified with a dephasing length.

## V. DISCUSSION

The possible dependence of  $M^*(r)$  at  $T = 0$  has been of interest as a means of monitoring anomalies in the

ground state<sup>9,13</sup> of metals. Previous studies proposed  $M^* \sim r^\mu$  with either<sup>9,15</sup> a small  $\mu$  or<sup>13</sup>  $\mu = 1.8$  or<sup>14</sup>  $\mu = 0$ . Instanton based arguments suggested<sup>13</sup> a  $M^*(r)$  dependence for  $\alpha r > 1$ .

With our revised values of  $N_T$  we were able to reach a reasonably large  $r$  only for weak coupling,  $\alpha = 0.019$ . For this coupling we observe saturation at  $3 < r < 20$ . Although we cannot strictly rule out  $\mu \neq 0$  at higher  $r$ , we find it highly unlikely that an  $r$  dependence will reappear at  $r > 20$ . We propose then  $\mu = 0$  at  $\alpha = 0.019$ , implying  $\mu = 0$  at all  $\alpha$  (if larger  $\alpha$  would show a  $\mu \neq 0$  it would imply an unlikely singular line in the  $\alpha, r$  plane). We propose then that  $\mu = 0$  for all  $\alpha$  at  $r \gg 1$  and that the effect of the environment is a mass renormalization, in agreement with the variational study<sup>14</sup>.

We have found temperature dependent length scales. For  $r \gtrsim 1$  we find  $T^{-1}$ , while for  $r \ll 1$  we find  $T^{-1/4}$ . We note that the same  $T$  dependence was found for dephasing lengths in a nonequilibrium study based on the purity of a reduced density matrix<sup>16</sup> for the dirty metal situation. A dephasing length was deduced<sup>16</sup> by comparing a dephasing rate with a mean level separation as a condition for coherence. It is remarkable that the agreement in these dephasing lengths is obtained in both regimes  $r \gtrsim 1$  and  $r \ll 1$  where the form of Eq. (2 ii) is valid for a dirty metal environment; the  $r \ll 1$  form is also valid for other realizations of a CL environment. We have therefore the intriguing observation that equilibrium scales can identify non-equilibrium dephasing length scales.

## APPENDIX A: PERTURBATION EXPANSION

Consider the action of a particle on a ring in presence of a dissipative environment and a flux  $\phi_x$  through the ring Eq. (1) with the dirty metal environment:

$$K(z) = 1 - [4r^2 \sin^2 \frac{z}{2} + 1]^{-1/2} = \sum_{n=1}^{\infty} a_n \sin^2(\frac{1}{2}nz); \quad \alpha = \frac{3}{8k_F^2 l^2} \quad (\text{A1})$$

For a low  $T$  expansion it is efficient to perform a duality transformation using the Poisson sum:

$$\sum_m g(m) = \int_{-\infty}^{\infty} d\phi \sum_p e^{2\pi i \phi p} g(\phi) \quad (\text{A2})$$

where the sums  $m, p$  run on all integers. Hence Eq. (1) becomes

$$\begin{aligned} Z &= Z_1 \int_{-\infty}^{\infty} d\phi \sum_p e^{2\pi i \phi(p + \phi_x) - \pi^2 t \phi^2} \times [1 - \\ &\alpha \sum_n a_n \int_0^\beta d\tau \int_0^\beta d\tau' \frac{\pi^2 T^2}{2 \sin^2[\pi T(\tau - \tau')]} (1 - \cos(2\pi n T \phi(\tau - \tau'))) \langle \cos[n(\theta(\tau) - \theta(\tau'))] \rangle_0] \end{aligned} \quad (\text{A3})$$

where  $t = 2MR^2T$ ,  $\beta = 1/T$ ,  $Z_1 = \int \mathcal{D}\theta \exp(-S_1\{\theta\})$  and the  $\langle \dots \rangle_0$  average is taken with respect to  $\exp(-S_1)$ , where

$$S_1\{\theta\} = \int_0^\beta d\tau \frac{1}{2} MR^2 \left( \frac{\partial \theta}{\partial \tau} \right)^2 \quad (\text{A4})$$

For a Gaussian average we have

$$\begin{aligned} \langle \cos[n(\theta(\tau) - \theta(\tau'))] \rangle_0 &= \exp[-\frac{1}{2}n^2 \langle (\theta(\tau) - \theta(\tau'))^2 \rangle_0] \\ &= \exp[-\frac{n^2}{\beta^2} \sum_\omega \langle |\theta(\omega)|^2 \rangle_0 (1 - \cos \omega(\tau - \tau'))] \\ &= \exp[-\frac{2n^2}{\beta^2 t} \sum_\omega \frac{1 - \cos \omega(\tau - \tau')}{\omega^2}] = e^{-n^2 |\tau - \tau'| / \beta t} \end{aligned} \quad (\text{A5})$$

where  $\theta(\tau) = \frac{1}{\beta} \sum_\omega e^{-i\omega\tau} \theta(\omega)$  and  $\omega$  are Matsubara frequencies  $\omega = 2\pi T \times \text{integer}$ .

For periodic functions we can change integration variables to  $\tau_1 = \tau - \tau'$ ,  $\tau_2 = \frac{1}{2}(\tau + \tau')$  with  $\int d\tau_2 = \beta$ , and  $|\tau_1|$  in (A5) is chosen in the range  $(-\beta/2, \beta/2)$  to allow for periodicity and continuity at  $\tau_1 = 0$ ; hence,

$$\begin{aligned} Z &= Z_1 \int_{-\infty}^{\infty} d\phi \sum_p e^{2\pi i \phi(p + \phi_x) - \pi^2 t \phi^2} \times \\ &\quad [1 - \beta\alpha \sum_n a_n \int_{-\beta/2}^{\beta/2} \frac{\pi^2 T^2}{2 \sin^2(\pi T \tau)} (1 - \cos(2\pi n T \phi \tau)) e^{-n^2 |\tau| / \beta t}] \end{aligned} \quad (\text{A6})$$

Integrating  $\phi$  we obtain

$$\begin{aligned} Z \sim & \sum_p [e^{-\frac{(p + \phi_x)^2}{t}} - \beta\alpha \sum_n a_n \int_0^{\beta/2} \frac{\pi^2 T^2}{\sin^2(\pi T \tau)} (e^{-\frac{(p + \phi_x)^2}{t}} - \frac{1}{2} e^{-\frac{(p + \phi_x - n T \tau)^2}{t} - \frac{n^2 |\tau|}{\beta t}} \\ & - \frac{1}{2} e^{-\frac{(p + \phi_x + n T \tau)^2}{t} - \frac{n^2 |\tau|}{\beta t}})] \equiv \sum_p e^{-\frac{(p + \phi_x)^2}{t}} (1 - \frac{\delta F}{T}) \end{aligned} \quad (\text{A7})$$

where the correction to the free energy  $\delta F$  is

$$\begin{aligned} \delta F = \alpha \sum_p \frac{e^{-\frac{(p + \phi_x)^2}{t}}}{\sum_{p'} e^{-\frac{(p' + \phi_x)^2}{t}}} \sum_n a_n \int_0^{\beta/2} \frac{\pi^2 T^2}{\sin^2(\pi T \tau)} [1 - \frac{1}{2} e^{2n \frac{T}{t} \tau (p + \phi_x) - n^2 \frac{T^2}{t} \tau^2 - n^2 \frac{T}{t} \tau} \\ - \frac{1}{2} e^{-2n \frac{T}{t} \tau (p + \phi_x) - n^2 \frac{T^2}{t} \tau^2 - n^2 \frac{T}{t} \tau}] \end{aligned} \quad (\text{A8})$$

where actually  $\frac{T}{t} = 1/2MR^2$ . At small  $\tau$  there are  $\int d\tau/\tau$  integrals and therefore a cutoff  $1/\omega_c$  is needed. At low temperatures  $t \ll 1$  one can retain only  $p = p' = 0$  and then the cutoff is not needed, as found below. Hence for  $t \ll 1$ ,

$$\delta F = \alpha \sum_n a_n \int_0^{\beta/2} \frac{\pi^2 T^2}{\sin^2(\pi T \tau)} [1 - e^{-n^2 \frac{T^2}{t} \tau^2 - n^2 \frac{T}{t} \tau} \cosh(2n\tau\phi_x T/t)] + O(e^{-1/t} \ln \omega_c T) \quad (\text{A9})$$

The effective mass  $M^*$  is defined in terms of the curvature, so that the 1st order correction is

$$\delta \frac{1}{M^* R^2} = \frac{\partial^2 \delta F}{\partial \phi_x^2} \Big|_0 = -\alpha \sum_n a_n \int_0^{\beta/2} \frac{\pi^2 T^2}{\sin^2(\pi T \tau)} (2n\tau T/t)^2 e^{-n^2 \frac{T^2}{t} \tau^2 - n^2 \frac{T}{t} \tau} \quad (\text{A10})$$

Note that there is no divergence at  $\tau = 0$ . The dominant integration range is  $\tau < t/Tn^2$  so that the 1st term in the exponent can be expanded; keeping terms to order  $t^2$  we obtain in terms of  $x = \tau n^2 / 2MR^2$ ,

$$\begin{aligned} \delta \frac{M}{M^*} &= -2\alpha \sum_n a_n \int_0^\infty (1 + \frac{\pi^2 t^2}{3n^4} x^2 - \frac{t}{n^2} x^2 + \frac{t^2}{2n^4} x^4 + \dots) e^{-x} dx \\ &= -2\alpha \sum_n a_n (1 - \frac{2t}{n^2} + (\frac{2\pi^2}{3} + 12) \frac{t^2}{n^4} + \dots) \end{aligned} \quad (\text{A11})$$

Hence to 1st order in  $t$

$$\frac{M}{M^*} = 1 - 2\alpha \sum_n a_n + 4t\alpha \sum_n \frac{a_n}{n^2} \quad (\text{A12})$$

At  $t = 0$  this result is consistent with Eq. 9 of Ref. 13.

The following sum rules are useful for evaluating these sums. Integrating Eq. (A1)  $\int_0^\pi dz$  we obtain:

$$\sum_{n=1}^{\infty} a_n = 2 - \frac{2}{\pi} \int_0^\pi \frac{dz}{\sqrt{4r^2 \sin^2 \frac{1}{2}z + 1}} \quad (\text{A13})$$

Fourier transform of Eq. (A1)

$$a_n = \frac{-4}{\pi} \int_0^\pi \left( 1 - \frac{1}{\sqrt{4r^2 \sin^2 \frac{1}{2}z + 1}} \right) \cos nz \, dz \quad (\text{A14})$$

and performing the  $n$  summation, we obtain

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} = \frac{4}{\pi} \int_0^\pi \frac{1}{\sqrt{4r^2 \sin^2 \frac{1}{2}z + 1}} \left( \frac{\pi^2}{6} - \frac{\pi z}{2} + \frac{z^2}{4} \right) dz. \quad (\text{A15})$$

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- <sup>1</sup> R. A. Webb, S. Washburn, C. P. Umbach, and R. B. Laibowitz, Phys. Rev. Lett. **54**, 2696 (1985).  
<sup>2</sup> E. M. Q. Jariwala, P. Mohanty, M. B. Ketchen, and R. A. Webb, Phys. Rev. Lett. **86**, 001594 (2001).  
<sup>3</sup> K. Yu. Arutyunov and T. T. Hongisto, Phys. Rev. B **70**, 064514 (2004).  
<sup>4</sup> I. Neder, M. Heiblum, Y. Levinson, D. Mahalu, and V. Umansky, Phys. Rev. Lett. **96**, 016804 (2006).  
<sup>5</sup> D. M. Harber, J. M. McGuirk, J. M. Obrecht and E. A. Cornell, J. Low Temp. Phys. **133**, 229 (2003).  
<sup>6</sup> M. P. A. Jones, C. J. Vale, D. Sahagun, B. V. Hall and E. A. Hinds, Phys. Rev. Lett. **91**, 080401 (2003).  
<sup>7</sup> Y. J. Lin, I. Teper, C. Chin and V. Vuletić, Phys. Rev. Lett. **92**, 050404 (2004).  
<sup>8</sup> P. Hyafil, J. Mozley, A. Perrin, J. TAILLEUR, G. Nogues, M. Brune, J.M. Raimond, and S. Haroche, Phys. Rev. Lett. **93**, 103001 (2004).  
<sup>9</sup> F. Guinea, Phys. Rev. B **65**, 205317 (2002).  
<sup>10</sup> W. Hofstetter and W. Zwerger, Phys. Rev. Lett. **78**, 3737 (1997).  
<sup>11</sup> C. P. Herrero, G. Schön and A. D. Zaikin, Phys. Rev. B **59**, 5728 (1999).  
<sup>12</sup> M. Büttiker and A. N. Jordan, Physica E (Amsterdam) **29**, 272 (2005).  
<sup>13</sup> D. S. Golubev, C. P. Herrero and A. D. Zaikin, Europhys. Lett. **63**, 426 (2003).  
<sup>14</sup> B. Horovitz and P. Le Doussal, Phys. Rev. B **74**, 073104 (2006).  
<sup>15</sup> The RG results of [9] are in fact consistent with  $\mu = 0$  [F. Guinea, private communication].  
<sup>16</sup> B. Horovitz and D. Cohen, Europhys. Lett. **81**, 30001 (2008); D. Cohen and B. Horovitz J. Phys. A: Math. Theor. **40**, 12281 (2007).  
<sup>17</sup> A Guide to Monte Carlo simulations in Statistical Physics, D. P. Landau and K. Binder, Cambridge University Press (2000)